INTERIOR ESTIMATES FOR SECOND-ORDER ELLIPTIC DIFFERENTIAL (OR FINITE-DIFFERENCE) EQUATIONS VIA THE MAXIMUM PRINCIPLE

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ABSTRACT
Second-order elliptic operators are transformed into second-order elliptic operators of a higher dimensionality acting on differences of functions. Applying the maximum principle to the resulting operators yields various a-priori pointwise estimates to difference-quotients of solutions of elliptic differential, as well as finite-difference, equations. We derive Schauder estimates, estimates for equations with discontinuous coefficients, and other estimates.

1. Simple example and introduction. In this paper a new method will be described for obtaining a-priori estimates for difference-quotients (and hence derivatives) of solutions to second-order elliptic differential equations. Such estimates (e.g., the Schauder estimates [10, 11]), play a basic role in the existence theory for linear, and in particular for non-linear, elliptic equations. In the literature (e.g. in [1], [2] and [7]) derivations of these estimates are based on potential theory and involve tools such as transformations of the independent variables and integral representations derived from the fundamental solution of the Laplace equation.

Our method is more elementary and can be illustrated by the following simple example:

Let $\phi(x) = \phi(x_1, x_2, \cdots, x_n)$ be a continuously twice-differentiable function on a bounded domain Ω in the real n-space E_n, and let it satisfy the equation $L\phi(x) = f(x)$, where L is a second-order elliptic operator with constant coefficients and $f(x)$ is a given function. Since L is elliptic, it satisfies the well-known maximum principle of E. Hopf [9], and it is therefore easy (see Section 3 below) to estimate $\| \phi \| = \sup_{x \in \Omega} | \phi(x) |$ in terms of $\| f \|$ and the boundary values of ϕ. Suppose, then, that we wish to estimate *derivatives* of ϕ, say $\partial\phi(0)/\partial x_1$, in terms of $\| \phi \|$, $\| f \|$ and d, where $2d$ is the distance from 0 (the origin) to the boundary of Ω. Our approach will be to view the difference

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$$\phi_1(x_1, x_2, \cdots, x_n; y_1) \equiv \tfrac{1}{2}[\phi(x_1 + y_1, x_2, \cdots, x_n) - \phi(x_1 - y_1, x_2, \cdots, x_n)]$$

as a function of the $n+1$ variables. This function is well-defined on the $n+1$-dimensional domain

$$\Omega_1 = \{(x_1, x_2, \cdots, x_n; y_1) | \quad x^2 < d^2, \ 0 < y_1 < d\},$$

where $x^2 = {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2$. Writing

$$L_1 = L - \nu \frac{\partial^2}{\partial {x_1}^2} + \nu \frac{\partial^2}{\partial {y_1}^2}, \qquad (\nu > 0),$$

we observe that, for sufficiently small ν, this new operator is elliptic in the $n+1$ variables, and satisfies

$$|L_1 \phi_1| = |L\phi_1| \leqq |f| \quad \text{in } \Omega_1 .$$

Hence, introducing the "comparison function"

$$\bar{\Phi}_1 = \frac{1}{2\nu} \| f \| (dy_1 - y_1^2) + \frac{1}{d^2} \| \phi \| \{x^2 + y_1^2 + C(dy_1 - y_1^2)\},$$

where C is a constant large enough to satisfy

$$2\nu C \geqq Lx^2 \;\; 2 + \nu \text{ throughout } \Omega$$

it is plain that

$$L_1 \bar{\Phi}_1 \leqq - \| f \| \leqq - |L_1 \phi_1| \text{ in } \Omega_1$$

and

$$\bar{\Phi}_1 \geqq |\phi_1| \text{ on the boundary } \partial\Omega_1 .$$

Thus $L_1(\bar{\Phi}_1 - \phi_1) \leqq 0$ in Ω_1 and $\bar{\Phi}_1 - \phi_1 \geqq 0$ on $\partial\Omega_1$, and hence, by the maximum-principle, $\bar{\Phi}_1 - \phi_1 \geqq 0$ throughout Ω_1. Similarly $\bar{\Phi}_1 + \phi_1 \geqq 0$ and it follows that $|\phi_1| \leqq \bar{\Phi}_1$. This implies in particular that

$$\begin{aligned} \tfrac{1}{2} | \phi(y_1, 0, \cdots, 0) - \phi(-y_1, 0, \cdots, 0) | &\leqq \bar{\Phi}_1(0, \cdots, 0; y_1) \\ &\leqq \frac{1}{2\nu} \| f \| dy_1 + \frac{1}{d^2} \| \phi \| (Cdy_1 + y_1^2). \end{aligned}$$

Dividing through by y_1 we get an estimate for a difference-*quotient* of ϕ, which, upon letting y_1 tend to zero, yields the desired estimate

$$\left| \frac{\partial \phi(0)}{\partial x_1} \right| \leqq \frac{d}{2\nu} \| f \| + \frac{C}{d} \| \phi \| .$$

(Note that it is essential in our argument that the intersection of a neighborhood of the origin ($x = 0,\ y_1 = 0$) with the hyperplane $\{y_1 = 0\}$ is included in $\partial\Omega_1$. On this hyperplane ϕ_1 vanishes; this enables us to construct a comparison-function $\bar{\Phi}_1$ such that $\bar{\Phi}_1(0,0,\cdots,0;\ y_1)/y_1$ is bounded.)

In this illustration, as in general, our method for estimating difference-quotients is based solely on the maximum principle, applied either to the given elliptic operator L or to some higher dimensional, derived elliptic operator, such as L_1 in the example.

This simple technique possesses the advantage of many elementary arguments, namely, a wider applicability; in fact, our method, unlike the alternative approaches mentioned earlier, is applicable not only to elliptic *differential* equations, but also to elliptic *finite-difference* equations. The reader can easily check this in the above example, upon replacing L and L_1 by corresponding finite-difference operators. Indeed, second-order finite-difference operators, like their differential counterparts, satisfy a maximum principle (not always in its naïve form, though; see [5]), and that is all we need for carrying our method through.

The need for a-priori estimates to solutions of elliptic *finite-difference* equations was the main motivation for our work(1). Such estimates can play a fundamental role in a *convergence theory for numerical solutions* to linear, and especially non-linear, elliptic equations. Furthermore, such finite-difference estimates can be used to demonstrate convergence of discrete approximations to a solution of a given elliptic differential equation *without making any a-priori assumptions about existence and smoothness of solutions.* This, in turn, may serve as an alternative approach to *prove* existence theorems for linear or non-linear elliptic differential equations.

In spite of this primal interest in finite-difference equations, in the present paper we only apply our technique to *differential* equations. This, however, is merely done to save in notation and simplify our vocabulary. Actually, every theorem and every proof can be immediately adapted, line by line, to the finite difference case. (Only the explicit "comparison functions" used in Sections 6–9 may, when discretized, need some minor modifications.)

In Section 2 we describe the higher-dimensional elliptic operator $L^{(q)}$ to be used

(1) In [4] we derived such estimates, but only for discrete operators whose main part is the Laplace operator. The method there was the "antisymmetrization" method, which is a very special case of the present "higher-dimensionality" method. For related results see also [3] and in particular [12], where interior estimates are established by means of discrete Fourier transforms and a discrete Sobolev inequality. The results in the present paper, however, are stronger, in the sense that we impose much weaker assumptions concerning the smoothness of the operators' coefficients. This weakness of assumptions is crucial for many applications.

in estimating q-order difference-quotients of solutions to a general second-order elliptic equation. For equations with constant coefficients an $(n+q)$-dimensional operator would do (as in the above example, where $q=1$), but for the general equation with variable coefficients $L^{(q)}$ is defined as an operator on $C^2(E_n^{q+1})$, i.e., it is $(q+1)n$-dimensional.

In Section 3 the maximum principle is reviewed. Applying this principle to $L^{(q)}$, we derive in Section 4 a general theorem that assesses q-order differences of solutions in terms of higher-dimensional comparison-functions. Various *interior estimates*, such as the *interior Schauder estimates* (Section 8) and Hölder-type estimates for equations with *discontinuous coefficients* (Section 6), are shown to be direct corollaries of that general theorem. A special treatment for equations in two variables and discontinuous coefficients is given in Section 9.

Other kinds of estimates are available by the same and similar methods. In a subsequent article we shall demonstrate the corresponding *near-the-boundary estimates*, again without resorting to coordinate-transformations or integral representations, thus keeping the results analogously obtainable in the finite-difference case. Our technique is also applicable to *parabolic equations*.

2. The basic identity. We first introduce some finite-difference notation. For any function F defined on some portion of the real n-space E_n we define

$$\delta(y)F(x) = \tfrac{1}{2}[F(x+y)-F(x-y)]$$

$$\mu(y)F(x) = \tfrac{1}{2}[F(x+y)+F(x-y)]$$

where $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)\in E_n$. Higher-order difference and mean operators are defined by

$$\delta_q(Y) = \delta_q(y^1,y^2,\cdots,y^q) = \delta(y^1)\delta(y^2)\cdots\delta(y^q)$$

and

$$\mu_q(Y) = \mu_q(y^1,y^2,\cdots,y^q) = \mu(y^1)\mu(y^2)\cdots\mu(y^q),$$

where $y^i=(y^i_1,y^i_2,\cdots,y^i_n)\in E_n$ and $Y=(y^1,y^2,\cdots,y^q)\in E_n^{\,q}$. The argument Y will often be omitted. We also set $\delta_0\equiv\mu_0\equiv 1$. Mixed products operators are introduced as follows:

$$\delta_{q-p}^{i_1,i_2,\cdots,i_p}(Y) = \mu_p(y^{i_1},y^{i_2},\cdots,y^{i_p})\cdot\delta_{q-p}(y^{i_{p+1}},y^{i_{p+2}},\cdots,y^{i_q})$$

and

$$\mu_{q-p}^{i_1,i_2,\cdots,i_p}(Y) = \delta_p(y^{i_1},y^{i_2},\cdots,y^{i_p})\cdot\mu_{q-p}(y^{i_{p+1}},y^{i_{p+2}},\cdots,y^{i_q})$$

where $(i_1, i_2, \cdots, i_p;\ i_{p+1}, i_{p+2}, \cdots, i_q)$ is any combination of $(1, 2, \cdots, q)$, i.e., a permutation for which $i_1 < i_2 < \cdots < i_p$ and $i_{p+1} < i_{p+2} < \cdots < i_q$. The following lemma, which is a discrete analogue of Leibnitz' rule, is easily proved by induction.

LEMMA 2.1. *If F and G are two functions defined on E_n (or on any subset of E_n such that $\delta_q(Y)F(x)$ and $\delta_q(Y)G(x)$ are meaningful) then*

$$\delta_q\{F\,G\} = \sum_{p=0}^{q} \eta_p^q[F;G]$$

where

$$\eta_p^q[F;G] = \sum \mu_{q-p}^{i_1,i_2,\cdots,i_p} F.\ \delta_{q-p}^{i_1,i_2,\cdots,i_p} G,$$

this later sum being carried out over all $\binom{q}{p}$ combinations

$$1 \leqq i_1 < i_2 < \cdots < i_p \leqq q.$$

Let us now consider a uniformly elliptic operator L defined by

$$L\phi(x) = \sum_{i,j=1}^{n} a_{ij}(x)\phi_{ij}(x) + \sum_{i=1}^{n} a_i(x)\phi_i(x) - a(x)\phi(x) \tag{2.1}$$

where $\phi_{ij}(x) = \dfrac{\partial^2\phi(x)}{\partial x_i \partial x_j}$ and $\phi_i(x) = \dfrac{\partial\phi(x)}{\partial x_i}$. The coefficients $a_{ij}(x) = a_{ji}(x)$, $a_i(x)$ and $a(x) \geqq 0$ are all bounded functions such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geqq \nu \sum^{n} \xi_i^2 \tag{2.2}$$

for all $(\xi_1, \cdots, \xi_n) \in E_n$ and every x in a certain domain $\Omega \subseteq E_n$, ν being a positive constant (the "constant of ellipticity") independent of either ξ or x. Denoting

$$\psi(x, Y) = \delta_q(Y)\phi(x),$$

$$\psi_{x_i} = \frac{\partial\psi}{\partial x_i}, \qquad \psi_{y_i^k} = \frac{\partial\psi}{\partial y_i^k}, \qquad \psi_{x_i y_j^k} = \frac{\partial\psi}{\partial x_i\,\partial y_j^k}$$

it is easy to see that

$$\begin{aligned} \psi_{x_i} &= \delta_q\phi_i \\ \psi_{y_i^k} &= \delta_{q-1}^k\ \phi_i \\ \psi_{x_i x_j} &= \psi_{y_i^k y_j^k} = \delta_q\phi_{ij} \\ \psi_{x_i y_j^k} &= \psi_{y_i^k x_j} = \delta_{q-1}^k\phi_{ij} \\ \psi_{y_i^k y_j^l} &= \psi_{y_i^l y_j^k} = \delta_{q-2}^{k,l}\phi_{ij}, \end{aligned} \tag{2.3}$$

where $1 \leqq k,\ l \leqq q$ and $k \neq l$. Applying Lemma 2.1 and relations (2.3) we get the following basic identity:

$$
\begin{aligned}
\delta_q(Y)L\phi(x) &= \sum_{i\,j} \delta_q\{a_{ij}\phi_{ij}\} + \sum_i \delta_q\{a_i\phi_i\} + \delta_q\{-a\phi\} \\
&= \sum_{i,j} \Bigg\{\Big(\mu_q a_{ij} - \sum_{k=1}^{q} b_{ij}^k\Big)\cdot\psi_{x_i x_j} + \sum_{k=1}^{q} b_{ij}^k \psi_{y_i^k y_j^k} \\
&\quad + \sum_{k=1}^{q} \mu_{q-1}^k a_{ij}\cdot(c_{ij}^k \psi_{x_i y_j^k} + (1-c_{ij}^k)\psi_{y_i^k x_j}) \\
&\quad + \sum_{1\leqq k<l\leqq q} \mu_{q-2}^{k,l}\, a_{ij}\cdot(d_{ij}^{kl}\psi_{y_i^k y_j^l} + (1-d_{ij}^{kl})\,\psi_{y_i^l y_j^k}) \qquad (2.4) \\
&\quad + \sum_{p=3}^{q} \eta_p^q[a_{ij};\phi_{ij}]\Bigg\} \\
&\quad + \sum_i \Big\{\mu_q a_i\cdot\psi_{x_i} + \sum_{k=1}^{q}\mu_{q-1}^k a_i\cdot\psi_{y_i^k} + \sum_{p=2}^{q}\eta_p^q[a_i;\phi_i]\Big\} \\
&\quad - \mu_q a\cdot\psi - \sum_{p=1}^{q}\eta_p^q[a;\phi].
\end{aligned}
$$

Here $b_{ij}^k = b_{ij}^k(x,Y)$, $c_{ij}^k = c_{ij}^{kl}(x,Y)$ and $d_{ij}^{kl} = d_{ij}^{kl}(x,Y)$ are arbitrary real functions, as yet at our disposal. We observe that (2.4) can be rewritten in the form

$$\delta_q\{L\phi\} = L^{(q)}(\delta_q\phi) + L_{q-1}\phi \qquad (2.5)$$

where

$$
\begin{aligned}
L_{q-1}\phi \equiv \sum_{i,j}\sum_{p=3}^{q}\eta_p^q[a_{ij};\phi_{ij}] + \sum_i\sum_{p=2}^{q}\eta_p^q[a_i;\phi_i] \qquad (2.6)\\
- \sum_{p=1}^{q}\eta_p^q[a;\phi].
\end{aligned}
$$

We call L_{q-1} the *residual operator*, and notice that it contains derivatives and differences of ϕ of orders not greater than $q-1$. Our main operator $L^{(q)}$ is $(q+1)n$-dimensional, i.e. it operates on functions defined on a certain part of E_n^{q+1}, the $(q+1)n$-dimensional Euclidean space whose coordinates are $x, y^1, y^2, \cdots, y^q$.

It is interesting to remark that in deriving the basic identity (2.4) we made no use of the ellipticity of L. This identity therefore holds for any second-order differential operator. It also holds, with obvious modifications, for second-order *difference* operators. Moreover, an identity of the form (2.5) is clearly obtainable for operators L of arbitrary order.

Here, however, we restrict our attention to uniformly elliptic operators L, in which case, for purposes indicated in the introduction and more fully explained below, we like $L^{(q)}$ to be itself elliptic. To that end the above undetermined functions are usually(2) chosen as follows:

$$c_{ij}^{k} \equiv d_{ij}^{kl} \equiv 1 \tag{2.7}$$

and

$$b_{ij}^{k} = A_k \delta_{ij} + A y_i^k y_j^k / |y^k|^2 \tag{2.8}$$

where δ_{ij} is the Kronecker Delta, A_k and A are non-negative functions to be specified later (in terms of the coefficients a_{ij} and their differences) and

$$|y^k|^2 = (y_1^k)^2 + \cdots + (y_n^k)^2 .$$

Thus we can now write

$$L^{(q)} = L_1^{(q)} + L_2^{(q)} + L_3^{(q)} + L_4^{(q)} \tag{2.9}$$

where

$$L_1^{(q)} = A \sum_{k=1}^{q} \sum_{i,j} \frac{y_i^k y_j^k}{|y^k|^2} \frac{\partial^2}{\partial y_i^k \partial y^k}$$

$$\begin{aligned} L_2^{(q)} = {} & A_0 \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2 + \sum_{k=1}^{q} A_k \sum_{i=1}^{n} \left(\frac{\partial}{\partial y_i^k}\right)^2 , \\ & + \sum_{k=1}^{q} \sum_{i,j} (\mu_{q-1}^{k} a_{ij}) \frac{\partial^2}{\partial x_i \partial y_j^k} \\ & + \sum_{1 \leq k < l \leq q} \sum_{i,j} (\mu_{q-2}^{k,l} a_{ij}) \frac{\partial^2}{\partial y_i^k \partial y_j^l} \end{aligned}$$

$$\begin{aligned} L_3^{(q)} = {} & \sum_{i,j} \left(\mu_q a_{ij} - A \sum_{k=1}^{q} \frac{y_i^k y_j^k}{|y^k|^2} \right) \frac{\partial^2}{\partial x_i \partial x_j} \\ & - \left(\sum_{k=0}^{q} A_k \right) \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2 , \end{aligned}$$

and

$$L_4^{(q)} = \sum_{i=1}^{n} (\mu_q a_i) \frac{\partial}{\partial x_i} + \sum_{k=1}^{q} \sum_{i=1}^{n} (\mu_{q-1}^{k} a_i) \frac{\partial}{\partial y_i^k} - \mu_q a .$$

(2) Unless otherwise explicitly stated. In Section 9, a different, more sophisticated choice is made.

Clearly, $L_1^{(q)}$ is semi-elliptic, i.e. its matrix of coefficients, at any point, is positive semi-definite. Also $L_3^{(q)}$ is semi-elliptic, provided that

$$qA + \sum_{k=0}^{q} A_k \leqq \nu, \tag{2.10}$$

where ν is the positive quantity in (2.2). Thus, to have $L^{(q)}$ [uniformly] elliptic, all we need is to make $L_2^{(q)}$[uniformly] elliptic, e.g. by making its matrix of coefficients [uniformly] diagonally dominant, that is, by requiring

$$A_0 - \tfrac{1}{2}\max_i \sum_{k=1}^{q} \sum_j \left|\mu_{q-1}^k a_{ij}\right| \geqq \eta \tag{2.11}$$

and

$$A_k - \tfrac{1}{2}\max_i\left\{\sum_{l\neq k} \sum_j \left|\mu_{q-2}^{k,l} a_{ij}\right| + \sum_j \left|\mu_{q-1}^k a_{ij}\right|\right\} \geqq \eta, \quad k = 1,2,\cdots,q. \tag{2.12}$$

where η is positive [and constant]. We have thus proved

THEOREM 2.1. *If $A, A_0, A_1, \cdots, A_q$ satisfy* (2.10–2.12) *with [constant] positive η, then $L^{(q)}$ is [uniformly] elliptic.*

Note that we could most easily satisfy (2.10–2.12) by taking $A \equiv 0$. This, however, would give us an elliptic operator $L^{(q)}$ which would be useless for our purposes. It becomes apparent below that $L_1^{(q)}$, which can be interpreted as

$$L_1^{(q)} = A \sum_{k=1}^{q} \frac{\partial^2}{\partial r_k{}^2}, \text{ where } r_k = |y^k|,$$

is the most "useful" part of $L^{(q)}$. We thus wish to keep A as large, and $A_0, A_1, \cdots, A_q$ as small, as possible.

3. Maximum and comparison principles. We now state the well-known maximum principle for uniformly elliptic operators L.

Maximum principle: *If a function u satisfies $Lu \geqq 0$ in Ω and has a maximum at an interior point of Ω then* $u \equiv$ const.

The proof of this theorem is quite elementary (see [6], page 326). Another way of stating it, which follows immediately by substituting $u = \pm\phi - \bar{\Phi}$, is the following

Comparison principle: *If $|L\phi| \leqq -L\bar{\Phi}$ in Ω and $|\phi| \leqq \bar{\Phi}$ on the boundary of Ω, then $|\phi| \leqq \bar{\Phi}$ throughout Ω.*

The comparison principle gives us an effective tool to estimate the solution ϕ of a Dirichlet problem for the equation $L\phi = f$. All we need to do is to construct a *comparison function* $\bar{\phi}$ which actually satisfies the above conditions.

Our purpose is to show that this tool is also effective in estimating difference-quotients of ϕ.

REMARK: The maximum principle, and hence also the comparison principle' obviously hold true even if L is only "*locally uniformly elliptic*", i.e., if each interior point of Ω has a neighborhood where L is uniformly elliptic. Thus, even if the positive quantities, ν in (2.2) and similarly η in (2.11–2.12), are not constant, but continuous, the corresponding operators still satisfy the maximum and the comparison principles.

4. Blueprint for estimating difference-quotients. Given a boundary value problem for the equation

$$L\phi(x) = f(x), \quad x \in \Omega, \tag{4.1}$$

with L as defined in (2.1–2.2), we can write, by (2.5),

$$L^{(q)}(\delta_q \phi) = f^{(q)} \tag{4.2}$$

where

$$f^{(q)}(x, Y) = \delta_q(Y)f(x) - L_{q-1}(x, Y)\phi(x). \tag{4.3}$$

Our blueprint for estimating difference-quotients of ϕ will be as follows: We first estimate $\max|\phi|$, e.g., by using the comparison principle as described in the previous section; then we estimate first-order difference-quotients of ϕ; then second-order difference-quotients; etc. Thus, in the process of assessing q-order difference-quotients we may assume that difference quotients of lower orders have already been estimated, and in particular, that an appropriate upper bound for $|f^{(q)}|$ has been obtained(3). We have thus enough information to construct a function $\bar{\psi} = \bar{\psi}(x, y^1, \cdots, y^q)$ for which

(3) We adopt this procedure since it serves best to illustrate our technique. Alternatively, one could use simple calculus considerations (see Section 2 in [7]) to estimate $|f^{(q)}|$. This estimate would have the form

$$\| f^{(q)} \| \leq \| f \| + \varepsilon \| \phi \|_q + C(\varepsilon, q) \| \phi \|_0,$$

where $\|\phi\|_k$ is some maximum norm of the k-order derivative of ϕ, ε is an arbitrary positive constant and $C(\varepsilon,q)$ is a constant depending only on the arguments shown. Since our method below essentially gives an estimate of the form

$$\| \phi \|_q \leqq C_1 \| \phi \|_0 + C_2 \| f^{(q)} \|,$$

one could, by selecting ε smaller than C_2^{-1}, immediately derive the desired estimate of $\|\phi\|_q$ in terms of $\|f\|$ and $\|\phi\|_0$, without having gone through any special analysis for orders less than q.

(4.4) $$-L^{(q)}\bar{\psi} \geqq |f^{(q)}| \text{ in a certain domain } \Omega^{(q)} \subset E_n^{q+1}$$

and

(4.5) $$\bar{\psi} \geqq |\delta_q\phi| \text{ on the boundary } \partial\Omega^{(q)}.$$

Subject to certain conditions on the coefficients of L and an accordingly proper choice of $A, A_0, A_1, \cdots, A_q$ (see Theorem 2.1) the $(q+1)n$-dimensional operator $L^{(q)}$ is (locally uniformly) elliptic in $\Omega^{(q)}$ and the comparison principle is therefore applicable to it, giving, via (4.2) and (4.4–4.5), the estimate

(4.6) $$|\delta_q\phi| \leqq \bar{\psi} \text{ throughout } \Omega^{(q)}.$$

Moreover, the domain $\Omega^{(q)}$ is chosen so that its boundary $\partial\Omega^{(q)}$ contains certain portions of the q subspaces $\{y^1 = 0\}, \cdots, \{y^q = 0\}$. On these subspaces $\delta_q\phi$ vanishes, and hence $\bar{\psi}$ (whose boundary values should satisfy (4.5)) may vanish there too. Thus the comparison function $\bar{\psi}$ can be constructed so that its modulus is small in the vicinity of these subspaces, e.g., $|\bar{\psi}| = 0(|y^1|\cdot|y^2|\cdots|y^q|)$, and therefore (4.6) yields, as desired, an appraisal for q-order *divided*-differences (or derivatives) of ϕ.

In the present paper we carry out such a program to obtain various "*interior estimates*."

An interior estimate to a difference-quotient of ϕ at some point $P \in \Omega$ is an estimate which depends on $\|\phi\| = \sup_{X\in\Omega}|\phi(x)|$ and on the distance d from P to the boundary $\partial\Omega$, but is otherwise independent of the boundary conditions.

With no loss of generality we may take the point P to be at the origin of E_n. We set

(4.7) $$\Omega^{(q)} = \{(x, y^1, \cdots, y^q) \mid 0 \leqq |x| < d_0, \quad 0 < |y^k| < d_1, \ k = 1, \cdots, q\}$$

to be the domain in E_n^{q+1} on which $\delta_q\phi$ is to be compared with $\bar{\psi}$. We require that the positive constants d_0 and d_1 satisfy

(4.8) $$d_0 + qd_1 \leqq d,$$

which ensures that $\delta_q\phi$ is indeed well-defined on $\Omega^{(q)}$. Thus $\Omega^{(q)}$ is a "holed" domain whose boundary can be desccribed as

(4.9) $$\partial\Omega^{(q)} = B_0^{(q)} \cup B_1^{(q)} \cup B_2^{(q)}$$

where

$$B_0^{(q)} = \partial\Omega^{(q)} \cap \bigcup_{k=1}^{q} \{y^k = 0\}$$

$$B_1^{(q)} = \partial\Omega^{(q)} \cap \bigcup_{k=1}^{q} \{|y^k| = d_1\}$$

$$B_2^{(q)} = \partial\Omega^{(q)} \cap \{|x| = d_0\}.$$

Since we shall be looking for a comparison function $\bar{\psi}$ which depend only upon the magnitudes, $|x|$ and $|y^k|$, of its coordinates, we shall also employ the "reduced" ($q+1$-dimensional) region

$$\Omega^{[q]} = \{(r_0, r_1, \cdots, r_q) \mid \ 0 \leqq r_0 < d_0; \ \ 0 < r_k < d_1, \ \ k = 1, \cdots, q\} \tag{4.10}$$

whose relation to $\Omega^{(q)}$ is evident. For any function $u = u(r_0, r_1, \cdots, r_q)$ in $C^2(\Omega^{[q]})$ we denote

$$u_k = \frac{\partial u}{\partial r_k}, \quad u_{kl} = \frac{\partial^2 u}{\partial r_k \partial r_l}, \quad k, l = 0, 1, \cdots, q,$$

and introduce the following "reduced" operator

$$\begin{aligned} L^{[q]}u = {} & \frac{\nu}{2q} \sum_{k=1}^{q} u_{kk} \\ & + \frac{nq}{2} \sum_{k=1}^{q} m_1(r_k)\left(u_{kk} + \frac{n-1}{r_k}u_k - u_{00} - \frac{n-1}{r_0}u_0\right) \\ & + n \sum_{k=1}^{q} m_1(r_k)\,|u_{0k}| + n \sum_{1 \leqq k < l \leqq q} m_2(r_k, r_l)\,|u_{kl}| \\ & + \left(n + \frac{1}{2}\right) m \left(|u_{00}| + \frac{1}{r_0}|u_0|\right) \\ & + n^{\frac{1}{2}}\bar{m}\,|u_0| + n^{\frac{1}{2}} \sum_{k=1}^{q} \bar{m}_1(r_k)\,|u_k|, \end{aligned} \tag{4.11}$$

where ν is the positive constant in (2.2) and the m's are bounds related to the coefficients of L in the neighborhood of the origin:

$$m = \sup_{|x| \leqq d_0 + qd_1} |a_{ij}(x)|, \tag{4.12}$$

$$m_1(r) = \sup_{\substack{|x| \leqq d_0 + (q-1)d_1 \\ |y| \leqq r}} |\delta(y)\, a_{ij}(x)|, \tag{4.13}$$

$$(4.14)\qquad m_2(r,s) = \sup_{\substack{|x|\leqq d_0+(q-2)d_1\\ |y|\leqq r, |z|\leqq s}} |\delta_2(y,z)a_{ij}(x)|,$$

$$(4.15)\qquad \bar{m} = \sup_{|x|\leqq d_0+qd_1} |a_i(x)|$$

$$(4.16)\qquad \bar{m}_1(r) = \sup_{\substack{|x|\leqq d_0+(q-1)d_1\\ |y|\leqq r}} |\delta(y)a_i(x)|,$$

each sup being also taken with respect to all $1 \leqq i,j \leqq n$. The role of this "reduced" operator $L^{[q]}$ becomes clear in the following theorem.

THEOREM 4.1. *(Differences Comparison Theorem). Let ϕ be a solution of the uniformly elliptic equation*

$$L\phi(x) = f(x), \qquad x \in \Omega,$$

and let $d_0 > 0$ and $d_1 > 0$ be selected (i.e., $\Omega^{(q)}$ and $\Omega^{[q]}$ be defined) so that (4.8) *is satisfied and also*

$$(4.17)\qquad m_1(d_1) \leqq \frac{\nu}{q(q+1)n}.$$

Suppose there exists a non-negative function $u = u(r_0, r_1, \cdots, r_q)$ in $C^2(\Omega^{[q]})$ for which

$$(4.18)\qquad -L^{[q]}u(|x|,|y^1|,\cdots,|y^q|) \geqq |f^{(q)}(x,Y)|, \quad (x,Y)\in\Omega^{(q)},$$

$$(4.19)\qquad u(|x|,|y^1|,\cdots,|y^q|) \geqq |\delta_q(Y)\phi(x)|, \quad (x,Y)\in B_1^{(q)} \cup B_2^{(q)}.$$

Then

$$(4.20)\qquad |\delta_q(Y)\phi(x)| \leqq u(|x|,|y^1|,\cdots,|y^q|) \text{ throughout } \Omega^{(q)}.$$

REMARK 1. Notice that the comparison function $u(|x|,|y^1|,\cdots,|y^q|)$ is allowed to vanish on $B_0^{(q)}$. In applying this theorem we will always try to choose u so that it also fulfills a requirement of the form

$$(4.21)\qquad u(0,r_1,\cdots,r_q) \leqq Cr_1r_2\cdots r_q, \quad C = \text{constant},$$

[or somewhat weaker requirements], so that (4.20) immediately yields an estimate for q-order difference-*quotients* (and hence derivatives), namely,

$$\frac{|\delta_q(Y)\phi(0)|}{|y^1|\cdot|y^2|\cdots|y^2|} \leqq C$$

[Or somewhat weaker estimates, e.g., an estimate for the Hölder coefficients of the $(q-1)$-order derivatives of ϕ at $x=0$.]

REMARK 2. We tacitly assumed that (4.17) can be satisfied by selecting d_1 sufficiently small. In other words, we restricted ourselves to equations whose coefficients may be discontinuous, but with only small oscillations.

Proof. We put

$$\begin{aligned} A &= \frac{\nu}{2q} - 2\eta(|y^1|,\cdots,|y^q|) \\ A_0 &= \frac{n}{2}\sum_{k=1}^{q} m_1(|y^k|) + \eta(|y^1|,\cdots,|y^q|), \\ A_k &= \frac{nq}{2}\, m_1(|y^k|) + \eta(|y^1|,\cdots,|y^q|), \qquad k=1,\cdots,q, \end{aligned}$$

where η is an arbitrary *positive* function in $C(\Omega^{[q]})$ which is small enough to ensure $A>0$. Owing to condition (4.17) these A's satisfy (2.10). It is also plain that they satisfy (2.11) and (2.12). Hence, by Theorem 2.1, $L^{(q)}$ is elliptic. Moreover, since η is positive and continuous, $L^{(q)}$ actually is locally-uniformly elliptic, and the comparison principle thus holds for it. As a comparison function we take

$$\bar{\psi}(x,y^1,\cdots,y^q) = u(|x|,|y^1|,\cdots,|y^q|) + \varepsilon(d^{\,p} - |y^1|^p)$$

where ε is any pre-assigned positive constant and $p>2$ is large enough to warrant

$$L^{(q)}|y^1|^p \geqq |y^1|^{p-2}\,. \tag{4.22}$$

Dropping hereinafter the arguments $(|x|,|y^1|,\cdots,|y^k|)$ associated with u and its derivatives, we first observe that

$$\frac{\partial u}{\partial y_i^k} = \frac{y_i^k}{|y^k|}u_k$$

and

$$\frac{\partial^2 u}{\partial y_i^k \partial y_j^l} = \frac{y_i^k y_j^l}{|y^k|\cdot|y^l|}\,u_{kl} + \frac{\delta_{kl}}{|y^k|}\left(\delta_{ij} - \frac{y_i^k y_j^k}{|y^k|^2}\right)u_k$$

for any $0 \leqq k,l \leqq q$. (We put $x \equiv y^0$). Substituting these relations into (2.9) and putting the result alongside with (4.11), we see that(4)

$$L^{(q)}u \leqq L^{[q]}u + \eta\, Tu$$

(4) In fact, the reduced operation $L^{[q]}$ has been devised precisely to meet this inequality. The small quantities η and ε are mere technical auxiliaries to insure *locally uniform* ellipticity.

where Tu is some linear combination, with constant coefficients, of u_{kk} and $u_k/|y^k|$, $(k=0,1,\cdots,q)$. Hence, by choosing the function η small enough and using (4.22) and then (4.18) and (4.2), we get

$$\begin{aligned} L^{(q)}\bar{\psi}(x,Y) &\leqq L^{[q]}u + \eta Tu - |y^1|^{p-2} \leqq L^{(q)}u \\ &\leqq -|f^{(q)}(x,Y)| \\ &= -|L^{(q)}\psi(x,Y)|, \qquad (x,Y)\in\Omega^{(q)}, \end{aligned} \tag{4.23}$$

where $\psi(x,Y)\equiv\delta_q(Y)\phi(x)$. To compare ψ and $\bar{\psi}$ on the boundary $\partial\Omega^{(q)}$ we observe that by (4.19), $|\psi|\leqq\bar{\psi}$ on $B_1^{(q)}$ and $B_2^{(q)}$. The same inequality also holds on $B_0^{(q)}$, since ψ identically vanishes there whereas $\bar{\psi}$ is non-negative. Thus by (4.9),

$$|\psi|\leqq\bar{\psi} \text{ on } \partial\Omega^{(q)}.$$

We can therefore use the comparison principle (Section 3) to conclude from (4.23) that $|\psi|\leqq\bar{\psi}$ throughout $\Omega^{(q)}$. Since this is true for any $\varepsilon>0$, we must actually have $|\psi|\leqq u$. Q.E.D.

5. Notations and remarks.

Denoting by d_x the distance from a point $x\in\Omega$ to the boundary $\partial\Omega$, and also denoting

$$\begin{aligned} r_k &= |y^k| \\ r_{kl} &= \min\{|y^k|,|y^l|\} \\ r_{123} &= \min\{|y^1|,|y^2|,|y^3|\}, \end{aligned}$$

we now compile a list of the various constants (bounds) that will be used in the next sections. the 'sup's in this list are to be taken with respect to all $x\in\Omega$, $x\pm y\in\Omega$, $1\leqq i\leqq n$ and $1\leqq j\leqq n$.

$$C_1 = \sup|a_{ij}(x)|$$

$$C_2 = \sup d_x\cdot|a_i(x)|$$

$$C_3 = \sup d_x^2\cdot|a(x)|$$

$$C_4 = \sup d_x^\alpha\cdot\frac{|\delta(y)a_{ij}(x)|}{|y|^\alpha}$$

$$C_5 = \max\left\{3, 3\left(\frac{2nC_4}{\nu}\right)^{1/\alpha}, 3\left(\frac{3n(n-1)C_4}{\beta(\beta+1)\nu}\right)^{1/\alpha}, \frac{18n^{\frac{1}{2}}C_2'}{\beta(\beta+1)\nu}\right\}$$

$$C_6 = \frac{3}{\beta(\beta+1)\nu}C_5^{\beta-1}$$

$$C_7 = \max\left\{4, 4\left(\frac{6nC_4}{\nu}\right)^{1/\alpha}, 4\left(\frac{48n(n-1)C_4}{\beta(\beta+1)\nu}\right)^{1/\alpha}, 4\left(\frac{24C_4}{\alpha\nu}\right)^{1/\alpha}, \frac{384n^{\frac12}C_2'}{\beta\beta(+1)\nu}\right\}$$

$$C_8 = \sup d_x^{\,1+\alpha}\,\frac{|\delta(y)a_i(x)|}{|y|^\alpha}$$

$$C_9 = \sup d_x^{\,2+\alpha}\,\frac{|\delta(y)\,a(x)|}{|y|^\alpha}$$

$$C_{10} = \max\left\{4, 4\left(\frac{48n(2n-1)C_4}{\beta(\beta+1)\nu}\right)^{1/\alpha}, \frac{384n^{\frac12}C_2'}{\beta(\beta+1)\nu}\right\}$$

$$C_{11} = \frac{6n(n-1)}{(\alpha-\beta)}\left(\frac{4}{C_{10}}\right)^\alpha C_4 + \frac{3nC_1'}{4(1-\beta)C_{10}} + \frac{(2n+1)C_1+n^{\frac12}C_2}{2C_{10}^{\,2}}$$

$$\bar{C}_{12} = \frac{nC_4}{(1+\beta)\nu}\cdot\max\left\{\frac{90(3n-1)}{\beta}, \frac{20n}{\beta(\alpha-\beta)}, \frac{18(9n-2)}{1-\alpha}\right\}$$

$$C_{12} = 5\cdot\max\left\{1, \bar{C}_{12}^{\,1/\alpha}, \left(\frac{180n^{\frac12}C_8}{\beta(\beta+1)\nu}\right)^{1/(1+\alpha)}, \left(\frac{108\,n^{\frac12}C_8}{(1-\alpha)(1+\beta)\nu}\right)^{1/(1+\alpha)}\right\}$$

$$K_1 = \sup|\phi(x)|$$

$$K_2 = \sup d_x^{\,2}\cdot|f(x)|$$

$$K_3 = 9K_2' + 18[(2n+1)C_1 + n^{\frac12}C_2]K_1' + 9C_3'K_1$$

$$K_4 = 2C_6K_3 + C_6[3^\alpha n(n-1)C_4C_5^{2-\alpha} + 6n^{\frac12}\,C_2'C_5]K_1'$$

$$K_5 = C_5^{\,-\beta}K_4 + C_5K_1'$$

$$K_6 = 16K_2'' + 16nC_2''K_5 + 32C_3'K_1' + 16C_3''K_2$$

$$K_7 = \frac{4}{\nu}[K_6 + 32(2n+1)C_1K_1'' + 32n^{\frac12}C_2K_1'']$$

$$K_8 = \max\left\{4C_7K_5, \frac{48n(n-1)4^\alpha C_4}{\beta(\beta+1)(\alpha-\beta)\nu C_7^{\,\alpha}}K_7\right\}$$

$$K_9 = \frac{4096}{\nu}[(4n+2)C_1 + n^{\frac12}C_2]K_1''$$

$$K_{10} = \frac{5000}{\nu}[(4n+2)C_1 + n^{\frac12}C_2]K_1'''$$

$$H_1 = \sup d_x^{\,2+\alpha}\cdot\frac{|\delta(y)f(x)|}{|y|^\alpha}$$

$$H_2 = 4^{2+\alpha}[H_1' + nC_8'K_5 + C_9'K_5 + C_9'K_1]$$

$$H_3 = \max\left\{4C_{10}K_5, \frac{16}{\beta(\beta+1)\nu C_{10}^{\,\alpha}}H_2, \frac{96C_{11}}{\beta(\beta+1)\nu}K_9\right\}$$

$$H_4 = 5^{2+\alpha}[H_1'' + n^2C_4''H_3 + \frac{3}{4}nC_8'H_3 + \frac{1}{3}C_9H_3 + n\,C_8''K_5 + \frac{3}{2}C_9'K_5 + C_9''K_1]$$

$$H_5 = \frac{54}{(1-\alpha)(1+\beta)v}\cdot 3^{(1-\alpha)/\beta}\,H_4$$

$$H_6 = \max\left\{\frac{50}{9}\,C_{12}H_3,\ \frac{480[(2n+1)C_1 + n^{1/2}C_2]}{\beta(1-\beta^2)vC_{12}}K_{10},\ \frac{120.5^\alpha(n^{\frac{1}{2}}C_8 + 2nC_4)}{\beta(\beta+1)(\alpha-\beta)vC_{12}{}^\alpha}K_{10}\right\}$$

Primes on constants refer to additional differencing in the corresponding expressions. For example

$$C_2' = \sup d_x\cdot|\delta(y)\,a_1(x)|,\quad C_2'' = \sup d_x\cdot|\delta_2(Y)a_1(x)|,$$

$$C_8' = \sup d_x{}^{1+\alpha}\cdot\ \frac{|\delta_2(y_1,y_2)\,a_1(x)|}{r_{12}{}^\alpha},$$

$$C_8'' = \sup dx^{1+\alpha}\cdot\ \frac{|\delta_3(y_1,y_2,y_3)a_1(x)|}{r_{123}{}^\alpha},\text{ etc.}$$

Since such differencing only decreases the numerical value of the constant ($C_2 \geqq C_2' \geqq C_2''$, etc.), one can always omit primes in the above list without invalidating any of the theorems below.

REMARK 1. In a first reading of Sections 6–8 below it is advisable not to pay attention to the full particulars of the above constants. It is sufficient to check that "large enough" constants satisfy our claims. Indeed, for many theoretical applications, the only interesting features of these constants are (i) that they are finite; (ii) that each C_μ depends solely on the coefficients a_{ij}, a_i and a; and (iii) that each K_μ and each H_μ could be written in the form

(5.1) $$K_\mu = C_\mu^1K_1 + C_\mu^2K_2$$

and

(5.2) $$H_\mu = C_\mu^3K_1 + C_\mu^4K_2 + C_\mu^5H_1,$$

respectively, where C_μ^p are again positive constants depending only on the coefficients a_{ij}, a_i and a.

For some applications, however, especially in numerical analysis, explicit knowledge of numerical values of upper-bounds is required. This explains the pains we took to compile the above list. We made, however, no attempt to find anything like "the lowest possible bounds". (Although best estimates are sometimes computable. Cf. e.g. in [4], the remark on page 486).

REMARK 2. If the reader does wish to examine our estimates in detail, he should note that in the following sections we always set $d_1 \leqq d_0 = d/(q+2)$. As a result, if $(x, y^1, \cdots, y^q) \in \Omega^{(q)}$ and $\bar{x} = x \pm y \pm \cdots \pm y^q$ then $d_{\bar{x}} \geqq d/(q+2)$. This implies the following relations:

$$|a_{ij}(\bar{x})| \leqq C_1 \tag{5.3}$$

$$|a_i(\bar{x})| \leqq \frac{q+2}{d} C_2 \tag{5.4}$$

$$|a(\bar{x})| \leqq \left(\frac{q+2}{d}\right)^2 C_3 \tag{5.5}$$

$$|f(\bar{x})| \leqq \left(\frac{q+2}{d}\right)^2 K_2. \tag{5.6}$$

Similarly, if $(x, y^1, \cdots, y^q) \in \Omega^{(q)}$ and we put $\hat{x} = x \pm y^1 \pm \cdots \pm y^{q-1}$ and $y = y^q$, then $d_{\hat{x} \pm y} \geqq d(/q+2)$ and consequently

$$\frac{|\delta(y) a_{ij}(\hat{x})|}{|y|^\alpha} \leqq \left(\frac{q+2}{d}\right)^\alpha C_4 \tag{5.7}$$

$$\frac{|\delta(y) a_i(\hat{x})|}{|y|^\alpha} \leqq \left(\frac{q+2}{d}\right)^{1+\alpha} C_8 \tag{5.8}$$

$$\frac{|\delta(y) a(\hat{x})|}{|y|^\alpha} \leqq \left(\frac{q+2}{d}\right)^{2+\alpha} C_9 \tag{5.9}$$

$$\frac{|\delta(y) f(\hat{x})|}{|y|^\alpha} \leqq \left(\frac{q+2}{d}\right)^{2+\alpha} H_1 \tag{5.10}$$

We shall not write down all the similar relations that apply to the primed constants, such as

$$|\delta(y) a_i(\hat{x})| \leqq \frac{q+2}{d} C_2'. \tag{5.4'}$$

For the m's introduced in (4.12–4.16) our relations (5.3, 5.4, 5.7, 5.8) yield

$$m \leqq C_1, \qquad \bar{m} \leqq \frac{q+2}{d} C_2, \tag{5.11–12}$$

$$m_1(r) \leqq C_1' \quad \text{or} \quad m_1(r) \leqq \left(\frac{q+2}{d}\right)^\alpha C_4 r^\alpha, \tag{5.13–14}$$

(5.15–16) $$m_2(r_1,r_2) \leqq C_1'' \quad \text{or} \quad m_2(r_1,r_2) \leqq \left(\frac{q+2}{d}\right)^{\alpha} C_4' r_{12}^{\alpha},$$

(5.17–18) $$\bar{m}_1(r) \leqq \frac{q+2}{d} C_2' \quad \text{or} \quad \bar{m}_1(r) \leqq \left(\frac{q+2}{d}\right)^{1+\alpha} C_8 r^{\alpha}.$$

6. Corollaries: Interior estimates for first-order difference quotients. Substituting $q = 1$, $d_0 = d/3$ and

$$u(r_0,r_1) = g(r_1) + K_1' r_0^2/d_0^2$$

in Theorem 4.1, and using (5.5, 5.6, 5.11, 5.12), we readily get

THEOREM 6.1. *Let ϕ be a solution of the uniformly elliptic equation*

$$L\phi(x) = f(x), \qquad x \in \Omega.$$

Let d be the distance from a point 0 (the origin, say) to $\partial\Omega$ and let $0 < d_1 \leqq d/3$ be selected so that

(6.1) $$m_1(d_1) \leqq \frac{\nu}{2n}.$$

Suppose further that there exists a non-negative function $g(r) \in C^2([0,d_1])$ for which

$$L^1 g(r) \equiv \left\{\frac{\nu}{2} + \frac{n}{2} m_1(r)\right\} g''(r) + \frac{n(n-1)m_1(r)}{2r} g'(r) + n^{\frac{1}{2}} \bar{m}_1(r) \,|\, g'(r)$$

(6.2) $$\leqq - d^{-2} K_3, \qquad (0 < r < d_1),$$

and

(6.3) $$g(d_1) \geqq K_1'.$$

Then

(6.4) $$|\delta(y)\phi(0)| \leqq g(|y|) \quad \text{for all} \quad |y| < d_1.$$

The actual form of the function g depends on the circumstances. Let us first consider the case where the coefficients a_{ij} may be *discontinuous, but with only small oscillations*. More precisely, we assume that for sufficiently small d_1, $(0 < d_1 \leqq d/3)$, (6.1) holds and also, for some $0 < \alpha < 1$,

$$\bar{C} = [(1-\alpha)\nu - n(n-2+\alpha)m_1(d_1) - 2n^{\frac{1}{2}}\bar{m}_1(d_1) \cdot d_1]^{-1} > 0.$$

Writing

$$\bar{K} = \max\left\{\frac{d^\alpha}{d_1^\alpha} K_1', \frac{2d_1^{2-\alpha}}{\alpha d^{2-\alpha}} \bar{C} K_3\right\}$$

it is easy to check that

$$g(r) = \bar{K} r^\alpha / d^\alpha$$

satisfies (6.2) and (6.3) and as a result, by (6.4),

$$d^\alpha \cdot \frac{|\delta(y)\phi(0)|}{|y|^\alpha} \leqq \bar{K}.$$

This proves

THEOREM 6.2. *If $L\phi = f$ in Ω, where f is bounded, and if the principal coefficients have small oscillations, namely,*

$$\sup_{x,y\,\in\Omega} |a_{ij}(x) - a_{ij}(y)| < \min\left\{\frac{v}{n}, \frac{2v}{n(n-2)}\right\},$$

then ϕ is Hölder-continuous in Ω. If, moreover, all the principal coefficients are continuous at some point $P \in \Omega$, then ϕ at P is Hölder-continuous with exponent α arbitrarily close to 1.

A stronger theorem for equations in two variables ($n = 2$) is proved in Section 9.

Various other results are as easily obtained by employing other forms of the function g. For instance, taking

$$g(r) = \tilde{K}\frac{r}{d}\log\frac{d}{r},$$

with $\tilde{K}$ sufficiently large, there follows

THEOREM 6.3. *If $L\phi = f$ in Ω, where f is bounded, and if the principal coefficients satisfy a condition of the form*

$$\sup_{x\pm y\in\Omega} |\delta(y)\, a_{ij}(x)| \leqq \tilde{C}\left(\log\frac{d}{|y|}\right)^{-1},$$

where $\tilde{C} < v/n(n-1)$ is a constant, then ϕ is "almost differentiable", i.e.,

$$d \cdot \frac{|\delta(y)\phi(0)|}{|y|} \leqq \tilde{K}\log\frac{d}{|y|}.$$

Next we consider the case where the principal coefficients a_{ij} are Hölder-continuous for some given exponent $0 < \alpha < 1$, and we introduce an auxiliary exponent $0 < \beta < \alpha$. We assume, in other words, that the constants $C_1, C_2, C_3,$

C_4, C_5, C_6, C_7 as well as the bounds $K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8$, (see Section 5) are all finite.

THEOREM 6.4. *If $L\phi = f$ in Ω, where f is bounded and the principal coefficients of L are Hölder-continuous, then,*

(6.5) $$d_x \cdot \frac{|\delta(y)\phi(x)|}{|y|} \leqq K_5, \qquad x \pm y \in \Omega.$$

In particular,

(6.6) $$d_x |\phi_i(x)| \leqq K_5, \qquad x \in \Omega.$$

To prove this theorem one has simply to show, taking $d_1 = d/C_5$ and using (5.14, 5.17), that

$$g(r) = \frac{r}{d_1} K_1' + \frac{d_1^\beta r - r^{1+\beta}}{d_1^{1+\beta}} K_4$$

satisfies (6.2–6.3). This implies (6.4) and hence (6.5).

7. Corollaries: Interior estimates for second-order difference-quotients. In Theorem 6.4 the "differentiability" of ϕ is stated. In fact, without additional assumptions we can show that ϕ is "almost twice differentiable". We prove it by applying Theorem 4.1 to the case $q = 2$.

To fix $\Omega^{(2)}$ we set $d_0 = d/4$ and require $d_1 \leqq d/4$. Consequently, for any $(x, Y) = (x, y^1, y^2) \in \Omega^{(2)}$ we have $d_{x \pm y^1 \pm y^2} \geqq d/4$, and hence, by Theorem 6.4,

(7.1) $$|\mu_2(Y)\phi_i(x)| \leqq \frac{4}{d} K_5,$$

(7.2) $$|\mu_1^k(Y)\phi(x)| \leqq \frac{2}{d} K_5 |y^k|,$$

and

(7.3) $$|\delta_2(Y)\phi(x)| \leqq \frac{2}{d} K_5 r_{12},$$

where we put $r_{12} = \min\{|y^1|, |y^2|\}$. Using (7.1) as well as (5.4–5.6), we deduce directly from (4.3) the estimate

(7.4) $$|f^{(2)}(x, Y)| \leqq K_6/d^2, \qquad (x, Y) \in \Omega^{(2)}.$$

By (7.3) we can also write

(7.5) $$|\delta_2(Y)\phi(x)| \leqq \frac{2K_5}{dd_1} |y^1| \cdot |y^2|, \qquad (x, Y) \in B_1^{(2)}.$$

Taking $d_1 = d/C_7$ (which entails (4.17)) and using (7.4–7.5) and (5.11 = 5.17), it is straightforward to show that the comparison function

$$u(r_0, r_1, r_2) = K_7 \frac{r_1 r_2}{d^2} \log \frac{2d_1}{r_1 + r_2} + K_8 \frac{r_1 r_2}{d^2} \left(1 - \frac{r_1^\beta + r_2^\beta}{4d_1^\beta}\right) + 16K_1'' \frac{r_0^2}{d^2} \tag{7.6}$$

satisfies (4.18–4.19). Theorem 4.1 (with $x = 0$ in (4.20) thus yields

THEOREM 7.1. *If $L\phi = f$ in Ω, where f is bounded and the principal coefficients of L are Hölder-continuous, then*

$$d_x^2 \frac{|\delta_2(y^1, y^2)\phi(x)|}{|y^1| \cdot |y^2|} \leqq K_7 \log \frac{2C_7^{-1} d_x}{|y^1| + |y^2|} + K_8, \qquad |y^1|, |y^2| \leqq \frac{d_x}{C_7}. \tag{7.7}$$

Under the conditions of Theorem 7.1, (7.7) is essentially the best possible estimate. Stronger results require stronger assumptions. We shall assume, in the rest of this section and in Section 8, that $f(x)$ as well as the coefficients of L are all Hölder-continuous, with exponent $0 < \alpha < 1$. More precisely, we assume that K_1, H_1, C_4, C_8 and C_9, and hence also all the other bounds C_μ, K_μ and H_μ introduced in Section 5, are finite. Again β is some auxiliary exponent, satisfying $0 < \beta < \alpha$.

Again we put $d_0 = d/4$ and require $d_1 < d/4$. By (7.1–7.2), (5.8–5.10) and (4.3) there now follows

$$d^{2+\alpha} |f^{(2)}(x, Y)| \leqq H_2 r_{12}^\alpha, \quad (x, Y) \in \Omega^{(2)}. \tag{7.8}$$

Taking $d_1 = d/C_{10}$ and noting (7.8), (7.5) and (5.11 = 5.17), it is a matter of straight computation to show that the function

$$u(r_0, r_1, r_2) = H_3 \frac{r_1 r_2}{d^2} \left(1 - \frac{r_1^\beta + r_2^\beta}{4d_1^\beta}\right) + K_9 \frac{r_0^2 r_1 r_2}{d^4} \log \frac{2d_1}{r_1 + r_2} + 256K_1'' \frac{r_0^4}{d^4}$$

satisfies (4.18–4.19). Hence, Theorems 4.1 (with $x = 0$ in (4.20)) and 6.4 give

THEOREM 7.2. *If $L\phi = f$ in Ω, where f and all the coefficients of L are Hölder-continuous, then*

$$d_x^2 \frac{|\delta_2(y^1, y^2)\phi(x)|}{|y^1| \cdot |y^2|} \leqq H_3, \qquad |y^1|, |y^2| \leqq \frac{1}{2} d_x. \tag{7.9}$$

8. Corollaries: Interior Schauder estimates. Taking up the case $q = 3$, we set $d_0 = d/5$, and require $d_1 \leqq d/5$. As a result, for any $(x, Y) = (x, y^1, y^2, y^3) \in \Omega^{(3)}$ we have $d_{x \pm y^1 \pm y^2 \pm y^3} \geqq d/5$, and therefore, by Theorem 7.2,

$$|\mu_3(Y)\phi_{ij}(x)| \leqq \frac{25}{d^2} H_3$$

$$|\mu_2^k(Y)\phi_i(x)| \leqq \frac{25}{4d^2} H_3 |y^k|,$$

$$|\mu_1^{k,l}(Y)\phi(x)| \leqq \frac{25}{9d^2} H_3 |y^k| \cdot |y^l|,$$

and by Theorem 6.4

$$|\mu_3(Y)\phi_i(x)| \leqq \frac{5}{d} K_5,$$

$$|\mu_2^k(Y)\phi(x)| \leqq \frac{5}{2d} K_5 |y^k|.$$

Using these relations together with (5.7–5.10), we conclude from (4.3) that

$$d^{2+\alpha}|f^{(3)}(x, Y)| \leqq H_4 r_{123}^{\alpha}, \quad (x, Y) \in \Omega^{(3)}. \tag{8.1}$$

By Theorem 7.2 we can also write

$$|\delta_3(Y)\phi(x)| \leqq \frac{25H_3}{9d^2 d_1} |y^1| \cdot |y^3| \cdot |y^2|, \quad (x, Y) \in B_1^{(3)}. \tag{8.2}$$

We now select $d_1 = d/C_{12}$, which satisfies (4.17). Using (8.1–8.2) and also (5.11–5.18), it is straightforward (although cumbersome, if constants are to be examined in detail) to check(5) that the comparison-function

$$\begin{aligned} u(r_0, r_1, r_2, r_3) = {} & H_5 \frac{r_1 r_2 r_3}{d^{2+\alpha}} (r_1^{\beta} + r_2^{\beta} + r_3^{\beta})^{-(1-\alpha)/\beta} \\ & + H_6 \frac{r_1 r_2 r_3}{d^3} \left(1 - \frac{r_1^{\beta} + r_2^{\beta} + r_3^{\beta}}{6d_1^{\beta}}\right) \\ & + 625 K_1''' \frac{r_0^4}{d^4} \\ & + K_{10} \frac{r_0^2}{d^2} \left[\sum_{1 \leqq l < k \leqq 3} \frac{r_k r_l}{d^2} \left(\log \frac{4d_1}{r_k + r_l} - \right.\right. \\ & \left.\left. \log 2 \cdot \frac{r_k^{\beta} + r_l^{\beta}}{2d_1^{\beta}} \right) \right] \end{aligned}$$

satisfies (4.18–4.19). Using (4.20) with $y = 0$, and also (7.9), we get

---

(5) In addition to basic inequalities, a simple instance of the Muirhead's inequality (see [8], page 44) is employed.

THEOREM 8.1. *If $L\phi = f$ in Ω, where f and the coefficients of L are all Hölder-continuous, then*

$$d_x^{2+\alpha} \frac{|\delta_3(y^1, y^2, y^3)\phi(x)|}{|y^1||y^2|\,|y^3|^\alpha} \leqq H_5 + \left(\frac{|y^3|}{d_x}\right)^{1-\alpha} H_6, |y^k|, |y^2|, |y^3| \leqq \frac{1}{3} d_x.$$

Noticing (5.2) it is clear that this theorem is equivalent to the well-known interior Schauder estimates.

9. Equations in two variables and bounded coefficients. In the case $n = 2$, and for any given two-dimensional vector $y=(y_1, y_2)$, the operator (2.1) can be rewritten in the form

$$L\phi(x) = \sum_{i,j=1}^{2} (\alpha_i\alpha_j + \bar{v}y_{ij})\phi_{ij}(x) + \sum_{i=1}^{2} a_i(x)\phi_i(x) - a(x)\phi(x), \quad x = (x_1,x_2) \in \Omega, \tag{9.1}$$

where $y_{ij} = y_i y_j/|y|^2$ and consequently α_1, α_2 and $\bar{v}$ are functions that depend also on y:

$$\alpha_i = \alpha_i(x,y), \quad \bar{v} = \bar{v}(x,y).$$

By (2.2) these functions must satisfy

$$\alpha_1^2 + \alpha_2^2 \geqq v \quad \text{and} \quad \bar{v} \geqq v. \tag{9.2}$$

We now introduce four other functions, $\theta_+(x,y)$, $\theta_-(x,y)$, $\theta^+(x,y)$ and $\theta^-(x,y)$, defined on $\Omega^{(1)}$ (see (4.7–4.8)), such that

$$\theta_+(x,y)[y_2\alpha_1(x+y,y) - y_1\alpha_2(x+y,y)] = \theta_-(x,y)[y_2\alpha_1(x-y,y) - y_1\alpha_2(x-y,y)], \tag{9.3}$$

$$\max\{|\theta_+(x,y)|,\ |\theta_-(x,y)|\} = 1, \tag{9.4}$$

$$|\theta^+(x,y)| = \max\{|\theta_+(x,y)|,\ \eta(x,y)\}, \quad \theta_+/\theta^+ \geqq 0,$$

and

$$|\theta^-(x,y)| = \max\{|\theta_-(x,y)|,\ \eta(x,y)\}, \quad \theta_-/\theta^- \geqq 0,$$

where $0 < \eta(x,y) < 1$ is a certain continuous function. We further define 12 more functions as follows

$$\alpha_i^\pm = \alpha_i^\pm(x,y) = \theta^\pm(x,y)\cdot\alpha_i(x \pm y, y), \quad (i = 1,2)$$

$$a_i^\pm = a_i^\pm(x,y) = [\theta^\pm(x,y)]^2 \cdot a_i(x \pm y), \quad (i = 1,2)$$

$$v^{\pm} = v^{\pm}(x,y) = [\theta^{\pm}(x,y)]^2 \cdot \bar{v}(x \pm y, y),$$
$$\psi = \psi(x,y) = \tfrac{1}{2}[\phi(x+y) - \phi(x-y)],$$

and

$$\beta = \beta(x,y) = \begin{cases} (\alpha_1^+ - \alpha_1^-)|y|/y_1 & \text{if } |y_1| \geqq |y_2| \\ (\alpha_2^+ - \alpha_2^-)|y|/y_2 & \text{if } |y_1| < |y_2|. \end{cases}$$

Note that, no matter which of the two alternatives in the last definition is selected, we must have, by (9.3),

$$\alpha_i^+ - \alpha_i^- = \frac{y_i}{|y|}\beta + \eta_i, \qquad (i = 1,2), \tag{9.5}$$

where we adopt the rule to denote by $\eta_k = \eta_k(x,y)$ any function such that $|\eta_k(x,y)|$ can be made arbitrarily small (that is, smaller than any pre-assigned positive function in $C(\Omega^{(1)})$) by letting $\eta(x,y)$ be small enough.

The following identity is readily verified by writing its both sides explictly in terms of ϕ, ϕ_i, ϕ_{ij}.

$$\tfrac{1}{2}[\theta^{+2}L\phi(x+y) - \theta^{-2}L\phi(x-y)] = L^{(1)}\psi(x,y) - \delta[\theta^2 a]\cdot\mu(y)\phi(x), \tag{9.6}$$

where

$$L^{(1)} = L_1^{(1)} + L_2^{(1)} + L_3^{(1)},$$

$$L_1^{(1)} = \frac{1}{4}\sum_{i,j=1}^{2}\Big\{(\alpha_i^+ + \alpha_i^-)(\alpha_j^+ + \alpha_j^-)\frac{\partial^2}{\partial x_i\partial x_j} + (\alpha_i^+ - \alpha_i^-)(\alpha_j^+ - \alpha_j^-)\frac{\partial^2}{\partial y_i\partial y_j}$$
$$+ (\alpha_i^+ - \alpha_i^-)(\alpha_j^+ + \alpha_j^-)\frac{\partial^2}{\partial y_i\partial x_j} + (\alpha_i^+ + \alpha_i^-)(\alpha_j^+ - \alpha_j^-)\frac{\partial^2}{\partial x_i\partial y_j}\Big\},$$

$$L_2^{(1)} = \frac{1}{4}\sum_{i,j=1}^{2}\Big\{(v^+ + v^-)y_{ij}\Big(\frac{\partial^2}{\partial x_i\partial x_j} + \frac{\partial^2}{\partial y_i\partial y_j}\Big)$$
$$+ (v^+ - v^-)y_{ij}\Big(\frac{\partial^2}{\partial y_i\partial x_j} + \frac{\partial^2}{\partial x_i\partial y_j}\Big)\Big\},$$

$$L_3^{(1)} = \sum_{i=1}^{2}\Big\{\mu[\theta^2 a_i]\frac{\partial}{\partial x_i} + \delta[\theta^2 a_i]\frac{\partial}{\partial y_i}\Big\} - \mu[\theta^2 a],$$

$$\delta[\theta^2 a] = \tfrac{1}{2}[\theta^{+2}a(x+y) - \theta^{-2}a(x-y)], \ \mu[\theta^2 a] = \tfrac{1}{2}[\theta^{+2}a(x+y) + \theta^{-2}a(x-y)].$$

It is clear that $L_1^{(1)}$ is semi-elliptic. Also, it follows from (9.2) that $v^{\pm} \geqq \eta^2 v$ and as a result $L_2^{(1)}$ is locally uniformly elliptic. Hence $L^{(1)}$ itself is locally uniformly elliptic. Furthermore, we have, for any $F \in C^2((0, d_1))$,

$$
\begin{aligned}
L_2^{(1)}F(|y|) &= \frac{v^+ + v^-}{4} \sum_{i,j=1}^{2} y_{ij}\left(y_{ij}F'' + (\delta_{ij} - y_{ij})\frac{F'}{|y|}\right) \\
&= \frac{v^+ + v^-}{4} F''(|y|),
\end{aligned} \tag{9.7}
$$

and similarly, by (9.5),

$$
L_1^{(1)}F(|y|) = \frac{\beta^2}{4}F''(|y|) + \eta_3 F''(|y|) + \eta_4 F'(|y|). \tag{9.8}
$$

We shall use these operators to prove

THEOREM 9.1. *If ϕ satisfies the equation*

$$
L\phi(x) = f(x), \qquad x \in \Omega,
$$

where the coefficients of L, as well as f and ϕ itself, are all bounded, namely,

$$
\begin{aligned}
&[\alpha_i(x)]^2 \leqq C_1, && \bar{v}(x) \leqq C_1, \\
&d_x \cdot |a_i(x)| \leqq C_2, && 0 \leqq d_x{}^2 \cdot a(x) \leqq C_3, \\
&|\phi(x)| \leqq K_1 \quad \text{and} && d_x{}^2 \cdot |f(x)| \leqq K_2,
\end{aligned}
$$

then ϕ is Lipschitz continuous and satisfies

$$
d_x \cdot \frac{|\delta(y)\phi(x)|}{|y|} \leqq C^{(1)}K_1 + C^{(2)}K_2, \qquad x \pm y \in \Omega, \tag{9.9}
$$

where

$$
\begin{aligned}
C^* &= \min\left[\frac{1}{3}, \frac{v}{18C_2}\right], \\
C^{(1)} &= C^{*-1} + \frac{36}{v}C^*(5C_1 + 4C_2 + C_3) + \frac{18}{v}C_2, \\
C^{(2)} &= \frac{36}{v}C^*.
\end{aligned}
$$

Proof. We shall estimate $\psi(x,y) = \delta(y)\phi(x)$ by comparing it to the function

$$
\bar{\psi}(x,y) = \frac{K_1}{d_0{}^2}|x|^2 + \frac{K}{d^2}(d_1|y| - |y|^2) + \frac{K_1}{d_1}|y|, \quad (x,y) \in \Omega^{(1)}
$$

where d is the distance from 0 (the origin of E_2) to $\partial\Omega$ and where d_0 and d_1 are the usual parameters of $\Omega^{(1)}$, for which we stipulate, as in Section 6, that

$$
d_1 \leqq d_0 = \frac{d}{3}. \tag{9.10}
$$

K is a constant, as yet at our disposal. It is clear from (4.9) that

$$\bar{\psi}(x,y) \geqq |\psi(x,y)|, \qquad (x,y) \in \partial\Omega^{(1)}.$$

Therefore, in order to use the comparison principle to show that $\bar{\psi} \geqq |\psi|$ throughout $\Omega^{(1)}$, it is sufficient to prove that

$$L^{(1)}\bar{\psi}(x,y) \leqq -|L^{(1)}\psi(x,y)| \quad , \quad (x,y) \in \Omega^{(1)}.$$

By (9.6), (9.4) and (9.10) it is in fact enough to show that

$$L^{(1)}\bar{\psi}(x,y) \leqq -\frac{9}{d^2}[C_3K_1 + K_2], \quad (x,y) \in \Omega^{(1)}. \tag{9.11}$$

Substituting the above explicit expression for $\bar{\psi}$ and applying (9.7–9.8), we find

$$L^{(1)}\bar{\psi} = -\frac{K}{2d^2}(\beta^2 + v^+ + v^-) + \eta_5 K + \frac{K_1}{2d_0^{\,2}}\left\{\sum_{i=1}^{2}(\alpha_i^+ + \alpha_i^-)^2 + v^+ + v^-\right\} + L_3^{(1)}\bar{\psi}$$

Therefore, to satisfy (9.11) it is sufficient to have

$$\frac{K}{2d^2}(\beta^2 + v^+ + v^-) \geqq \frac{9}{d^2}[C_3K_1 + K_2] + \eta_5 K + \frac{5K_1}{d_0^{\,2}}C_1 + \frac{3C_2}{d}\left[\frac{4}{d_0}K_1 + \frac{3}{2}\left(\frac{K}{d^2}d_1 + \frac{K_1}{d_1}\right)\right].$$

Or, in view of (9.2), (9.4) and (9.10), it is sufficient that

$$\left(v - 9C_2\frac{d_1}{d} + \eta_6\right)K \geqq \left(90C_1 + 72C_2 + 18C_3 + 9C_2\frac{d}{d_1}\right)K_1 + 18K_2.$$

This last inequality is satisfied by taking

$$d_1 = C^*d \tag{9.12}$$

and

$$K = \frac{18}{v}(10C_1 + 8C_2 + 2C_3 + C_2/C^*)K_1 + \frac{36}{v}K_2 + \eta^*, \tag{9.13}$$

where η^* is a positive constant that can be made arbitrarily small by letting $\eta(x,y)$ be small enough.

Thus, with K and d_1 defined as in (9.12–9.13) we have $|\psi(x,y)| \leqq \bar{\psi}(x,y)$ throughout $\Omega^{(1)}$. In particular for $x = 0$ we get

$$|\psi(0,y)| \leqq \left(\frac{Kd_1}{d^2} + \frac{K_1}{d_1}\right) \cdot |y|, \qquad (|y| \leqq d_1),$$

or, by (9.12),

$$d \cdot \frac{|\delta(y)\phi(0)|}{|y|} \leqq C^*K + C^{*-1}K_1, \qquad (|y| \leqq C^*d).$$

This holds for any positive η^* and therefore also for $\eta^* = 0$. For $|y| \geqq C^*d$ the last inequality is self-evident. This inequality is thus equivalent to (9.9). Q.E.D.

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